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*Supplementary material*

SUPPLEMENTARY MATERIAL TO

# Computation of Zagreb and atom–bond connectivity indices of certain families of dendrimers by using automorphism group action

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## CONSIDERATIONS OF ZAGREB GROUP INDICES

It was Randić who presented in 1975 a bond-additive topological index called branching index as a descriptor for describing molecular branching.[1] The Randić index was defined as follows:

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$$

Later, this index was renamed as connectivity index and generalized to connectivity indices of various orders, such as the zero-*th* order, the first order connectivity index (second Zagreb index) and of higher orders. The families of the Zagreb indices are applied to investigate the complexity of molecules, chirality, *Z*/*E*-isomerism and hetero-systems. Moreover, the Zagreb indices revealed a probable applicability for developing multi-linear regression models and thus have great importance in QSPR/QSAR modeling. The first and second Zagreb indices of a graph *G* are defined as:

$$M_1(G) = \sum_{v \in U(G)} (d_v)^2 = \sum_{e=uv \in E(G)} (d_u + d_v)$$

$$M_2(G) = \sum_{e=uv \in E(G)} (d_u \times d_v)$$

The first and second Zagreb co-indices of a graph *G* are defined as:

$$\bar{M}_1(G) = \sum_{e=uv \notin E(G)} (d_u + d_v)$$

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$$\bar{M}_2(G) = \sum_{e=uv\notin E(G)} (d_u \times d_v)$$

where $d_u$ and $d_v$ denote the degree of the vertex $u$ and vertex $v$, respectively.

The first and second Zagreb polynomials for a graph $G$ and in variable $x$ are given as:

$$Zg_1(G,x) = \sum_{e=uv\in E(G)} x^{(d_u+d_v)}$$

$$Zg_2(G,x) = \sum_{e=uv\in E(G)} x^{(d_u\times d_v)}$$

The first and second modified Zagreb indices of a graph $G$ are defined as:

$$^{m}M_1(G) = \sum_{v\in U(G)} (d_v)^{-2} = \sum_{e=uv\in E(G)} (d_u + d_v)^{-1}$$

$$^{m}M_2(G) = \sum_{e=uv\in E(G)} (d_u \times d_v)^{-1}$$

The first and second variable Zagreb indices of a graph $G$ are defined as:

$$^{\upsilon}M_1(G) = \sum_{v\in U(G)} (d_v)^{2\upsilon} = \sum_{e=uv\in E(G)} (d_u + d_v)^{\upsilon}$$

$$^{\upsilon}M_2(G) = \sum_{e=uv\in E(G)} (d_u \times d_v)^{\upsilon}$$

for

$$\upsilon = 1, \qquad {}^{\upsilon}M_r(G) = M_r(G)$$

$$\upsilon = -1, \qquad {}^{\upsilon}M_r(G) = {}^{m}M_r(G)$$

and

$$\upsilon = \frac{-1}{2}, \qquad {}^{\upsilon}M_1(G) = \chi(G)$$

where $r$=1, 2.

In 2013, Shirdel, Rezapour and Sayadi[2] introduced a distance-based Zagreb index named hyper-Zagreb index that is given by:

$$HM(G) = \sum_{e=uv\in E(G)} (d_u + d_v)^2$$

In the augmented Zagreb index of a graph, $G$ is defined as:[3]

$$AZI(G) = \sum_{uv\in E(G)} \left(\frac{d_u \times d_v}{d_u + d_v - 2}\right)^3$$

*Proof for Theorem 1*

It is easy to see that $d_{v_{01}} = 2 = d_{v_{02}}$ , $d_{v_{(n+1)0}} = 1$ $d_{v_{it}} = 2 + \delta_{0t}$, where $1 \le i \le n$ and $0 \le t \le 3$. Now, by using Automorphism group action on the vertices and edges of $G[n]$ and Lemma 1, one obtains:

$$M_1(G[n]) = |I_1|(d_{v_{01}})^2 + |I_2|(d_{v_{02}})^2 + \sum_{i=1}^{n}\sum_{t=0}^{3}|I_{it}|(d_{v_{it}})^2 + 2^{n+1}(d_{v_{(n+1)0}})^2 =$$

$$= 17 \times 2^{n+2} - 50,$$

$$M_2(G[n]) = |S_1|(d_{v_{10}} d_{v_{01}}) + |S_2|(d_{v_{01}} d_{v_{02}}) + |S_3|(d_{v_{02}} d_{v_{03}}) +$$

$$+ \sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}|(d_{v_{i(t-1)}} d_{v_{it}}) + \sum_{i=1}^{n}|N_i|(d_{v_{i3}} d_{v_{(i+1)0}}) =$$

$$= 9 \times 2^{n+3} - 56,$$

$$Z_{g_1}(G,x) = \sum_{uv \in E(G)} x^{d_u + d_v} =$$

$$= |S_1| x^{(d_{v10} + d_{v01})} + |S_2| x^{(d_{v01} + d_{v02})} + |S_3| x^{(d_{v02} + d_{v03})} +$$

$$+ \sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}| x^{(d_{vi(t-1)} + d_{vit})} + \sum_{i=1}^{n}|N_i| x^{(d_{vi3} + d_{v(i+1)0})} =$$

$$= (3 \times 2^{n+1} - 6)x^5 + (2^{n+3} - 5)x^4 + 2^{n+1}x^3,$$

$$Z_{g_2}(G,x) = \sum_{uv \in E(G)} x^{d_u d_v} =$$

$$= |S_1| x^{(d_{v10} d_{v01})} + |S_2| x^{(d_{v01} d_{v02})} + |S_3| x^{(d_{v02} d_{v03})} +$$

$$+ \sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}| x^{(d_{vi(t-1)} d_{vit})} + \sum_{i=1}^{n}|N_i| x^{(d_{vi3} d_{v(i+1)0})} =$$

$$= (3 \times 2^{n+1} - 6)x^6 + (2^{n+3} - 5)x^4 + 2^{n+1}x^2.$$

*Proof for Theorem 2*

Since in the first and second Zagreb co-indices all those pairs of vertices are considered which are not adjacent with each other, one can write:

$$\overline{M}_1(G[n]) = M_1(K_n) - M_1(G[n]) =$$

$$= n(n-1)^2 - \sum_{i=1}^{k}|E_i|(d_{x_{j-1}} + d_{x_j}) =$$

$$= n(n-1)^2 - [\,|S_1|(d_{v_{10}} + d_{v_{01}}) + |S_2|(d_{v_{01}} + d_{v_{02}}) + |S_3|(d_{v_{02}} + d_{v_{03}})$$

$$+\sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}|(d_{v_{i(t-1)}} + d_{v_{it}}) + \sum_{i=1}^{n}|N_i|(d_{v_{i3}} + d_{v_{(i+1)0}})] =$$

$$= (16\times 2^n - 10)(16\times 2^n - 11)^2 - (17\times 2^{n+2} - 50) =$$

$$= (16\times 2^n - 10)(256\times 2^{2n} + 121 - 352\times 2^n) - (17\times 2^{n+2} - 50) =$$

$$= 4096\times 2^{3n} - 8192\times 2^{2n} + 5388\times 2^n - 1160\,,$$

$$\bar{M}_2(G[n]) = n(n-1)^2 - \sum_{i=1}^{k}|E_i|(d_{x_{j-1}} d_{x_j}) =$$

$$= n(n-1)^2 - [|S_1|(d_{v_{10}} d_{v_{01}}) + |S_2|(d_{v_{01}} d_{v_{02}}) + |S_3|(d_{v_{02}} d_{v_{03}}) +$$

$$+\sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}|(d_{v_{i(t-1)}} d_{v_{it}}) + \sum_{i=1}^{n}|N_i|(d_{v_{i3}} d_{v_{(i+1)0}})] =$$

$$= (16\times 2^n - 10)(16\times 2^n - 11)^2 - (9\times 2^{n+3} - 56) =$$

$$= 4096\times 2^{3n} + 1936\times 2^n - 5632\times 2^{2n} - 2560\times 2^{2n} -$$

$$-1210 + 3520\times 2^n - 72\times 2^n + 56 =$$

$$= 4096\times 2^{3n} - 8192\times 2^{2n} + 5384\times 2^n - 1154\,,$$

where $n = 16\times 2^n - 10$.

*Proof for Theorem 3*

$$^{\upsilon}M_1(G[n]) = |I_1|(d_{v_{01}})^{2\upsilon} + |I_2|(d_{v_{02}})^{2\upsilon} +$$

$$+\sum_{i=1}^{n}\sum_{t=0}^{3}|I_{it}|(d_{v_{it}})^{2\upsilon} + 2^{n+1}(d_{v_{(n+1)0}})^{2\upsilon} =$$

$$= 2(2^{2\upsilon}) + 2(2^{2\upsilon}) + \sum_{i=1}^{n}\sum_{t=0}^{3} 2^{i+1-\delta_{0t}}(2+\delta_{0t})^{2\upsilon} + 2^{n+1}(1)^{2\upsilon} =$$

$$= 4(2)^{2\upsilon} + \sum_{i=1}^{n}(2^i\times 3^{2\upsilon} + 2^{i+1}\times 2^{2\upsilon} + 2^{i+1}\times 2^{2\upsilon} + 2^{i+1}\times 2^{2\upsilon}) + 2^{n+1} =$$

$$= 4(2)^{2\upsilon} + 2(2^n - 1)(9^{\upsilon} + 62^{2\upsilon}) + 2^{n+1} =$$

$$= -2\{4\times 2^{2\upsilon} + 3^{2\upsilon}\} + 2^{n+1}\{1 + 3^{2\upsilon} + 6\times 2^{2\upsilon}\}\,.$$

$$^{\upsilon}M_2(G[n]) = |S_1|(d_{v_{10}} d_{v_{01}})^{\upsilon} + |S_2|(d_{v_{01}} d_{v_{02}})^{\upsilon} + |S_3|(d_{v_{02}} d_{v_{03}})^{\upsilon} +$$

$$+\sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}|(d_{v_{i(t-1)}} d_{v_{it}})^{\upsilon} + \sum_{i=1}^{n}|N_i|(d_{v_{i3}} d_{v_{(i+1)0}})^{\upsilon} =$$

$$=2(6)^{\upsilon}+2(4)^{\upsilon}+(4)^{\upsilon}+\sum_{i=1}^{n}2^{i+1}(6^{\upsilon}+4^{\upsilon}+4^{\upsilon})+\sum_{i=1}^{n-1}2^{i+1}6^{\upsilon}+2^{n+1}(2)^{\upsilon}=$$
$$=2(6)^{\upsilon}+3(4)^{\upsilon}+4(2^{n}-1)(6^{\upsilon}+2\times4^{\upsilon})+4(2^{n-1}-1)6^{\upsilon}+2^{n+1}(2)^{\upsilon}=$$
$$=-6\times6^{h}-5\times4^{\upsilon}+2^{n+1}\{3\times6^{\upsilon}+4\times4^{\upsilon}+2^{\upsilon}\}.$$

*Proof for Theorem 4*

$$HM(G[n])=|S_1|(d_{v_{10}}+d_{v_{01}})^2+|S_2|(d_{v_{01}}+d_{v_{02}})^2+$$
$$+|S_3|(d_{v_{02}}+d_{v_{03}})^2+\sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}|(d_{v_{i(t-1)}}+d_{v_{it}})^2+$$
$$+\sum_{i=1}^{n}|N_i|(d_{v_{i3}}+d_{v_{(i+1)0}})^2=$$
$$=98+114\sum_{i=1}^{n}2^{i}+50\sum_{i=1}^{n-1}2^{i}+18.2^{n}=37\times2^{n+3}-230,$$
$$AZI(G[n])=|S_1|(\frac{d_{v_{10}}d_{v_{01}}}{d_{v_{10}}+d_{v_{01}}-2})^3+|S_2|(\frac{d_{v_{01}}d_{v_{02}}}{d_{v_{01}}+d_{v_{02}}-2})^3+$$
$$+|S_3|(\frac{d_{v_{02}}d_{v_{03}}}{d_{v_{02}}+d_{v_{03}}-2})^3++\sum_{i=1}^{n}\sum_{t=1}^{3}|M_{it}|(\frac{d_{v_{i(t-1)}}d_{v_{it}}}{d_{v_{i(t-1)}}+d_{v_{it}}})^3+$$
$$+\sum_{i=1}^{n}|N_i|(\frac{d_{v_{i3}}d_{v_{(i+1)0}}}{d_{v_{i3}}+d_{v_{(i+1)0}}-2})^3=$$
$$=2(2)^3+2(2)^3+(2)^3+\sum_{i=1}^{n}2^{i+1}\{(2)^3+(2)^3+(2)^3\}+$$
$$+\sum_{i=1}^{n-1}2^{i+1}(2)^3+2^{n+1}(2)^3=$$
$$=2^{n+7}-88.$$

*Proof for Theorem 5*

It is easy to see that $d_{v_{01}}=2=d_{v_{02}}$, $d_{v_{(n+1)0}}=1$ and $d_{v_{it}}=2+\delta_{0t}$ where $1\leq$ $\leq i\leq n$ and $0\leq t\leq 6$. Now, by using Automorphism group action on the vertices and edges of $H[n]$ and Lemma 1, one obtains:

$$M_1(H[n])=|I|(d_{v_{01}})^2+\sum_{i=1}^{n}\sum_{t=0}^{6}|I_{it}|(d_{v_{it}})^2+2^{n+1}(d_{v_{(n+1)0}})^2$$

$$= 29 \times 2^{n+2} - 106,$$

$$M_2(H[n]) = |S_1|(d_{v_{10}} d_{v_{01}}) + |S_2|(d_{v_{01}} d_{v_{02}}) +$$

$$+ \sum_{i=1}^{n} \sum_{t=1}^{6} |M_{it}|(d_{v_{i(t-1)}} d_{v_{it}}) + \sum_{i=1}^{n} |N_i|(d_{v_{i6}} d_{v_{(i+1)0}}) =$$

$$= 2(3.2) + (2.2) + \sum_{i=1}^{n} 2^{i+1}\{(d_{v_{i0}} d_{v_{i1}}) + (d_{v_{i1}} d_{v_{i2}}) + (d_{v_{i2}} d_{v_{i3}}) + (d_{v_{i3}} d_{v_{i4}}) +$$

$$+ (d_{v_{i4}} d_{v_{i5}}) + (d_{v_{i5}} d_{v_{i6}})\} + \sum_{i=1}^{n-1} 2^{i+1}(d_{v_{i6}} d_{v_{(i+1)0}}) + 2^{n+1}(d_{v_{n6}} d_{v_{(n+1)0}}) =$$

$$= 15 \times 2^{n+3} - 112,$$

$$Z_{g1}(H, x) = \sum_{uv \in E(G)} x^{d_u + d_v} =$$

$$= |S_1| x^{(d_{v10} + d_{v01})} + |S_2| x^{(d_{v01} + d_{v02})} + \sum_{i=1}^{n} \sum_{t=1}^{6} |M_{it}| x^{(d_{vi(t-1)} + d_{vit})} +$$

$$+ \sum_{i=1}^{n} |N_i| x^{(d_{vi6} + d_{v(i+1)0})} =$$

$$= (3 \times 2^{n+1} - 6) x^5 + (5 \times 2^{n+2} - 19) x^4 + 2^{n+1} x^3,$$

$$Z_{g2}(H, x) = \sum_{uv \in E(G)} x^{d_u d_v} =$$

$$= |S_1| x^{(d_{v10} d_{v01})} + |S_2| x^{(d_{v01} d_{v02})} + \sum_{i=1}^{n} \sum_{t=1}^{6} |M_{it}| x^{(d_{vi(t-1)} d_{vit})} +$$

$$+ \sum_{i=1}^{n} |N_i| x^{(d_{vi6} d_{v(i+1)0})} =$$

$$= (3 \times 2^{n+1} - 6) x^6 + (5 \times 2^{n+2} - 19) x^4 + 2^{n+1} x^2.$$

*Proof for Theorem 6*

$$\bar{M}_1(H[n]) = n(n-1)^2 - \sum_{i=1}^{k} |E_i|(d_{x_{j-1}} + d_{x_j}) =$$

$$= n(n-1)^2 - [|S_1|(d_{v_{10}} + d_{v_{01}}) + |S_2|(d_{v_{01}} + d_{v_{02}}) +$$

$$+ \sum_{i=1}^{n} \sum_{t=1}^{6} |M_{it}|(d_{v_{i(t-1)}} + d_{v_{it}}) + \sum_{i=1}^{n} |N_i|(d_{v_{i6}} + d_{v_{(i+1)0}})] =$$

$$=(28\times 2^{n}-24)(28\times 2^{n}-25)^{2}-(29.2^{n+2}-106)=$$
$$=21952\times 2^{3n}-58016\times 2^{2n}+50984\times 2^{n}-14894,$$
$$\bar{M}_2(H[n])=n(n-1)^{2}-\sum_{i=1}^{k}|E_i|(d_{x_{j-1}}d_{x_j})=$$
$$=n(n-1)^{2}-[|S_1|(d_{v_{10}}d_{v_{01}})+|S_2|(d_{v_{01}}d_{v_{02}})+$$
$$+\sum_{i=1}^{n}\sum_{t=1}^{6}|M_{it}|(d_{v_{i(t-1)}}d_{v_{it}})+\sum_{i=1}^{n}|N_i|(d_{v_{i6}}d_{v_{(i+1)0}})]=$$
$$=(28\times 2^{n}-24)(28\times 2^{n}-25)^{2}-(15\times 2^{n+3}-112)=$$
$$=21952\times 2^{3n}-58016\times 2^{2n}+50980\times 2^{n}-14888,$$

where $n=28\times 2^{n}-24$.

*Proof for Theorem 7*

$$^{\upsilon}M_1(H[n])=|I|(d_{v_{01}})^{2\upsilon}+\sum_{i=1}^{n}\sum_{t=0}^{6}|I_{it}|(d_{v_{it}})^{2\upsilon}+2^{n+1}(d_{v_{(n+1)0}})^{2\upsilon}=$$
$$=-2\{11\times 2^{2\upsilon}+3^{2\upsilon}\}+2^{n+1}\{1+3^{2\upsilon}+12\times 2^{2\upsilon}\},$$
$$^{\upsilon}M_2(H[n])=|S_1|(d_{v_{10}}d_{v_{01}})^{\upsilon}+|S_2|(d_{v_{01}}d_{v_{02}})^{\upsilon}+$$
$$+\sum_{i=1}^{n}\sum_{t=1}^{6}|M_{it}|(d_{v_{i(t-1)}}d_{v_{it}})^{\upsilon}+\sum_{i=1}^{n}|N_i|(d_{v_{i6}}d_{v_{(i+1)0}})^{\upsilon}=$$
$$=-6\times 6^{\upsilon}-19\times 4^{\upsilon}+2^{n+1}\{3\times 6^{\upsilon}+10\times 4^{\upsilon}+2^{\upsilon}\}.$$

*Proof for Theorem 8*

$$HM(H[n])=|S_1|(d_{v_{10}}+d_{v_{01}})^{2}+|S_2|(d_{v_{01}}+d_{v_{02}})^{2}+$$
$$+\sum_{i=1}^{n}\sum_{t=1}^{6}|M_{it}|(d_{v_{i(t-1)}}+d_{v_{it}})^{2}+\sum_{i=1}^{n}|N_i|(d_{v_{i6}}+d_{v_{(i+1)0}})^{2}=$$
$$=66+210\sum_{i=1}^{n}2^{i}+50\sum_{i=1}^{n-1}2^{i}+18.2^{n}=$$
$$=463\times 2^{n}-404,$$

$$AZI(H[n]) = |S_1|\left(\frac{d_{v_{10}}d_{v_{01}}}{d_{v_{10}}+d_{v_{01}}-2}\right)^3 + |S_2|\left(\frac{d_{v_{01}}d_{v_{02}}}{d_{v_{01}}+d_{v_{02}}-2}\right)^3 +$$

$$+\sum_{i=1}^{n}\sum_{t=1}^{6}|M_{it}|\left(\frac{d_{v_{i(t-1)}}d_{v_{it}}}{d_{v_{i(t-1)}}+d_{v_{it}}}\right)^3 + \sum_{i=1}^{n}|N_i|\left(\frac{d_{v_{i6}}d_{v_{(i+1)0}}}{d_{v_{i3}}+d_{v_{(i+1)0}}-2}\right)^3 =$$

$$= 2(2)^3 + (2)^3 + \sum_{i=1}^{n}2^{i+1}\{(2)^3+(2)^3+(2)^3+(2)^3+(2)^3+(2)^3\}$$

$$+\sum_{i=1}^{n-1}2^{i+1}(2)^3 + 2^{n+1}(2)^3 =$$

$$= 7\times 2^{n+5} - 200\,.$$

*Proof for Theorem 9*

$$ABC_1(G[n]) = |S_1|\sqrt{\frac{d_{v_{10}}+d_{v_{01}}-2}{d_{v_{10}}d_{v_{01}}}} + |S_2|\sqrt{\frac{d_{v_{01}}+d_{v_{02}}-2}{d_{v_{01}}d_{v_{02}}}} +$$

$$+|S_3|\sqrt{\frac{d_{v_{02}}+d_{v_{03}}-2}{d_{v_{02}}d_{v_{03}}}} + \sum_{i=1}^{n}2^{i+1}\sum_{t=1}^{3}\sqrt{\frac{d_{v_{i(t-1)}}+d_{v_{it}}-2}{d_{v_{i(t-1)}}d_{v_{it}}}} +$$

$$+\sum_{i=1}^{n-1}2^{i+1}\sqrt{\frac{d_{v_{i3}}+d_{v_{(i+1)0}}-2}{d_{v_{i3}}d_{v_{(i+1)0}}}} + 2^{n+1}\sqrt{\frac{d_{v_{n3}}+d_{v_{(n+1)0}}-2}{d_{v_{n3}}d_{v_{(n+1)0}}}} =$$

$$= \frac{5}{\sqrt{2}} + \sum_{i=1}^{n}2^{i+1}\left(\frac{3}{\sqrt{2}}\right) + \sum_{i=1}^{n-1}2^{i+1}\left(\frac{1}{\sqrt{2}}\right) + 2^{n+1}\left(\frac{1}{\sqrt{2}}\right) =$$

$$= \frac{-11}{\sqrt{2}} + 2^n\times\frac{16}{\sqrt{2}}\,.$$

*Proof for Theorem 10*

If $uv$ is an edge in $G$ then $m_u$ is the number of vertices closer to $u$ than to $v$. As $v_{01}v_{02}\in E(G)$, the number of vertices closer to $v_{01}$ than to $v_{02}$ are:

$$m_{v_{01}} = 2+4(2)+4(2^2)+\ldots+4(2^n) = 2+4(2+2^2+\ldots+2^n) =$$

$$= 2+8(2^n-1) = 8\times 2^n - 6.$$

Similarly, $m_{v_{02}} = 8\times 2^n - 5$ and $m_{v_{it}} = 8\times 2^{n-i+\delta_{0t}} - (3+t+4\delta_{0t})$, where $1 \le$ $\le i \le n$ and $0 \le t \le 3$, and $m_{v_{(i+1)0}} = 8\times 2^{n-i} - 7$. Now, by using Automorphism group action on the vertices and edges of $G[n]$ and Lemma 2, one obtains:

$$ABC_2(G[n]) = |S_1|\sqrt{\frac{m_{v_{10}}+m_{v_{01}}-2}{m_{v_{10}}m_{v_{01}}}} + |S_2|\sqrt{\frac{m_{v_{01}}+m_{v_{02}}-2}{m_{v_{01}}m_{v_{02}}}} +$$

$$+|S_3|\sqrt{\frac{m_{v_{02}}+m_{v_{03}}-2}{m_{v_{02}}m_{v_{03}}}} + \sum_{i=1}^{n} 2^{i+1} \sum_{t=1}^{3} \sqrt{\frac{m_{v_{i(t-1)}}+m_{v_{it}}-2}{m_{v_{i(t-1)}}m_{v_{it}}}} +$$

$$+\sum_{i=1}^{n-1} 2^{i+1} \sqrt{\frac{m_{v_{i3}}+m_{v_{(i+1)0}}-2}{m_{v_{i3}}m_{v_{(i+1)0}}}} + 2^{n+1}\sqrt{\frac{m_{v_{n3}}+m_{v_{(n+1)0}}-2}{m_{v_{n3}}m_{v_{(n+1)0}}}} =$$

$$= \alpha_1[\frac{2}{\sqrt{\beta_1}} + \frac{2}{\sqrt{\beta_1+3}} + \frac{1}{\sqrt{\beta_1+4}} + \sum_{i=1}^{n}\{\frac{1}{\sqrt{\gamma_1-16(2^{n-i})-64(2^n)+3}}$$

$$+\frac{1}{\sqrt{\gamma_1-80(2^n)+4}} + \frac{1}{\sqrt{\gamma_1+16(2^{n-i})-96(2^n)+3}} +$$

$$+\frac{1}{\sqrt{\gamma_1+32(2^{n-i})-112(2^n)}}\}].$$

*Proof for Theorem 11*

As $S_{v_{01}} = 2+3+2 = 7$, similarly, $S_{v_{02}} = 6$, $S_{v_{i0}} = 9$, $S_{v_{i1}} = 7$, $S_{v_{i2}} = 6$, $1 \le i \le n$, $S_{v_{is}} = 7$, $1 \le i \le n-1$, $S_{v_{ns}} = 5$ and $S_{v_{(n+1)0}} = 3$, $S_{v_{it}} = 6 + 3\delta_{0t} + \delta_{1t} + \delta_{3t}$ where $1 \le i \le n-1$ and $0 \le t \le 3$.

$$= \left\{\sqrt{\frac{5}{18}} - 6\sqrt{\frac{11}{42}} - 6\sqrt{\frac{14}{63}}\right\} + 2^n\left\{6\sqrt{\frac{14}{63}} + 6\sqrt{\frac{11}{42}} + 2\sqrt{\frac{9}{30}} + 2\sqrt{\frac{3}{4}}\right\}$$

*Proof*:

As $\epsilon_{v_{02}} = \tau_1 = \epsilon_{v_{04}}, \epsilon_{v_{01}} = \tau_1 + 1 = \epsilon_{v_{03}}, \epsilon_{v(n+1)0} = \tau_1 + 6$ and $\epsilon_{v_{it}} = \tau_1 + 4i + t - 2$, where $1 \le i \le n$ and $0 \le t \le 3$. Here $\tau_1$ is defined[2] as $\tau_1 = 4n+3$.

*Proof for Theorem 12*

$$ABC_5(G[n]) = |S_1|\sqrt{\frac{\epsilon_{v_{10}}+\epsilon_{v_{01}}-2}{\epsilon_{v_{10}}\epsilon_{v_{01}}}} + |S_2|\sqrt{\frac{\epsilon_{v_{01}}+\epsilon_{v_{02}}-2}{\epsilon_{v_{01}}\epsilon_{v_{02}}}} +$$

$$+|S_3|\sqrt{\frac{\epsilon_{v_{02}}+\epsilon_{v_{03}}-2}{\epsilon_{v_{02}}\epsilon_{v_{03}}}} + \sum_{i=1}^{n} 2^{i+1} \sum_{t=1}^{3} \sqrt{\frac{\epsilon_{v_{i(t-1)}}+\epsilon_{v_{it}}-2}{\epsilon_{v_{i(t-1)}}\epsilon_{v_{it}}}} +$$

$$+\sum_{i=1}^{n-1} 2^{i+1} \sqrt{\frac{\epsilon_{v_{i3}}+\epsilon_{v_{(i+1)0}}-2}{\epsilon_{v_{i3}}\epsilon_{v_{(i+1)0}}}} + 2^{n+1}\sqrt{\frac{\epsilon_{v_{n3}}+\epsilon_{v_{(n+1)0}}-2}{\epsilon_{v_{n3}}\epsilon_{v_{(n+1)0}}}} =$$

$$=2\sqrt{\frac{2\tau_1+1}{(\tau_1+2)(\tau_1+1)}}+2\sqrt{\frac{2\tau_1-1}{\tau_1(\tau_1+1)}}+\frac{\sqrt{2\tau_1-2}}{\tau_1}+$$

$$+\sum_{i=1}^{n}2^{i+1}\{\sqrt{\frac{2\tau_1+8i+1}{(\tau_1+4i+1)(\tau_1+4i+2)}}+\sum_{t=1}^{3}\sqrt{\frac{2\tau_1+8i+2t-8}{(\tau_1+4i+t-2)(\tau_1+4i+t-3)}}\}=$$

$$=2\sqrt{\frac{8n+7}{(4n+5)(4n+4)}}+2\sqrt{\frac{8n+5}{(4n+3)(4n+4)}}+\frac{\sqrt{8n+4}}{4n+3}+$$

$$+\sum_{i=1}^{n}2^{i+1}\{\sqrt{\frac{8n+8i+7}{(4n+4i+4)(4n+4i+5)}}+\sum_{t=1}^{3}\sqrt{\frac{8n+8i+2t-2}{(4n+4i+t+1)(4n+4i+t)}}\}.$$

*Proof for Theorem 13*

$$ABC_1(H[n])=|S_1|\sqrt{\frac{d_{v_{10}}+d_{v_{01}}-2}{d_{v_{10}}d_{v_{01}}}}+|S_2|\sqrt{\frac{d_{v_{01}}+d_{v_{02}}-2}{d_{v_{01}}d_{v_{02}}}}+$$

$$+\sum_{i=1}^{n}2^{i+1}\sum_{t=1}^{6}\sqrt{\frac{d_{v_{i(t-1)}}+d_{v_{it}}-2}{d_{v_{i(t-1)}}d_{v_{it}}}}+\sum_{i=1}^{n-1}2^{i+1}\sqrt{\frac{d_{v_{i6}}+d_{v_{(i+1)0}}-2}{d_{v_{i6}}d_{v_{(i+1)0}}}}+$$

$$+2^{n+1}\sqrt{\frac{d_{v_{n6}}+d_{v_{(n+1)0}}-2}{d_{v_{n6}}d_{v_{(n+1)0}}}}=$$

$$=\frac{3}{\sqrt{2}}+\sum_{i=1}^{n}2^{i+1}(\frac{6}{\sqrt{2}})+\sum_{i=1}^{n-1}2^{i+1}(\frac{1}{\sqrt{2}})+2^{n+1}(\frac{1}{\sqrt{2}})=$$

$$=\frac{-25}{\sqrt{2}}+2^{n}\times\frac{28}{\sqrt{2}}$$

*Proof for Theorem 14*

It is noted that $m_{v_{01}}=14\times2^{n}-12, m_{v_{02}}=14\times2^{n}-11$ and $m_{v_{it}}=14\times2^{n-i+\delta_{0t}}-(6+t+7\delta_{0t})$ where $1\le i\le n$ and $0\le t\le 6$ and $m_{v_{(i+1)0}}=14\times2^{n-i}-10$. Now by using Automorphism group action on the vertices and edges of $H[n]$ and Lemma 2, one obtains:

$$ABC_2(H[n])=|S_1|\sqrt{\frac{m_{v_{10}}+m_{v_{01}}-2}{m_{v_{10}}m_{v_{01}}}}+|S_2|\sqrt{\frac{m_{v_{01}}+m_{v_{02}}-2}{m_{v_{01}}m_{v_{02}}}}+$$

$$+\sum_{i=1}^{n}2^{i+1}\sum_{t=1}^{6}\sqrt{\frac{m_{v_{i(t-1)}}+m_{v_{it}}-2}{m_{v_{i(t-1)}}m_{v_{it}}}}+\sum_{i=1}^{n-1}2^{i+1}\sqrt{\frac{m_{v_{i3}}+m_{v_{(i+1)0}}-2}{m_{v_{i3}}m_{v_{(i+1)0}}}}+$$

$$+2^{n+1}\sqrt{\frac{m_{v_{n6}}+m_{v_{(n+1)0}}-2}{m_{v_{n6}}m_{v_{(n+1)0}}}}=$$

$$=\alpha_2[\frac{2}{\sqrt{\beta_2}}+\frac{1}{\sqrt{\beta_2+1}}+\sum_{i=1}^{n}\{\frac{1}{\sqrt{\gamma_2-140(2^{n-i})-196(2^n)}}+$$

$$+\frac{1}{\sqrt{\gamma_2-112(2^{n-i})-224(2^n)+9}}+\frac{1}{\sqrt{\gamma_2-84(2^{n-i})-252(2^n)+16}}+$$

$$+\frac{2}{\sqrt{\gamma_2-56(2^{n-i})-280(2^n)+21}}+\frac{1}{\sqrt{\gamma_2-28(2^{n-i})-308(2^n)+24}}+$$

$$+\frac{1}{\sqrt{\gamma_2-336(2^n)+25}}\}].$$

*Proof for Theorem 15*

$S_{v_{01}}=7,\ S_{v_{i0}}=9,\ S_{v_{n1}}=7,\ S_{v_{i2}}=S_{v_{i3}}=S_{v_{i4}}=S_{v_{i5}}=6,\ 1\le i\le n,\ S_{v_{i6}}=7,$ $1\le i\le n-1,\ S_{v_{n6}}=5$ and $S_{v_{(n+1)o}}=3.\ S_{v_{it}}=6+\delta_{1t}+\delta_{2t}+\delta_{3t},$

where $1 \le i \le n-1$ and $0 \le t \le 6$.

$$ABC_4(H[n])=|S_1|\sqrt{\frac{S_{v_{10}}+S_{v_{01}}-2}{S_{v_{10}}S_{v_{01}}}}+|S_2|\sqrt{\frac{S_{v_{02}}+S_{v_{01}}-2}{S_{v_{02}}S_{v_{01}}}}+$$

$$+\sum_{i=1}^{n}2^{i+1}\sum_{t=1}^{6}\sqrt{\frac{S_{v_{i(t-1)}}+S_{v_{it}}-2}{S_{v_{i(t-1)}}S_{v_{it}}}}+\sum_{i=1}^{n-1}2^{i+1}\sqrt{\frac{S_{v_{i(6)}}+S_{v_{(i+1)0}}-2}{S_{v_{i(6)}}S_{v_{(i+1)0}}}}+$$

$$+2^{n+1}\sqrt{\frac{S_{v_{n(6)}}+S_{v_{(n+1)0}}-2}{S_{v_{n(6)}}S_{v_{(n+1)0}}}}=$$

$$=\left\{-2\sqrt{\frac{14}{63}}+12\sqrt{\frac{12}{49}}-8\sqrt{\frac{11}{42}}-12\sqrt{\frac{5}{18}}\right\}+$$

$$+2^n\left\{8\sqrt{\frac{14}{63}}+6\sqrt{\frac{11}{42}}+12\sqrt{\frac{5}{18}}+2\sqrt{\frac{2}{5}}\right\}$$

*Proof for Theorem 16*

As $\epsilon_{v_{01}}=\tau_2=\epsilon_{v_{02}}$, $\epsilon_{v_{(n+1)0}}=\tau_2+7n+1$ and $\epsilon_{v_{it}}=\tau_2+7i+t-6$, where $1\le$ $\le i\le n$ and $0\le t\le 6$. Here, $\tau_2=7n+2$.

$$ABC_5(H[n]) = |S_1|\sqrt{\frac{\epsilon_{v_{10}}+\epsilon_{v_{01}}-2}{\epsilon_{v_{10}}\epsilon_{v_{01}}}} + |S_2|\sqrt{\frac{\epsilon_{v_{01}}+\epsilon_{v_{02}}-2}{\epsilon_{v_{01}}\epsilon_{v_{02}}}} +$$

$$+\sum_{i=1}^{n}2^{i+1}\sum_{t=1}^{6}\sqrt{\frac{\epsilon_{v_{i(t-1)}}+\epsilon_{v_{it}}-2}{\epsilon_{v_{i(t-1)}}\epsilon_{v_{it}}}} + \sum_{i=1}^{n-1}2^{i+1}\sqrt{\frac{\epsilon_{v_{i3}}+\epsilon_{v_{(i+1)0}}-2}{\epsilon_{v_{i3}}\epsilon_{v_{(i+1)0}}}} +$$

$$+2^{n+1}\sqrt{\frac{\epsilon_{v_{n6}}+\epsilon_{v_{(n+1)0}}-2}{\epsilon_{v_{n6}}\epsilon_{v_{(n+1)0}}}}$$

$$= 2\sqrt{\frac{2\tau_2-1}{\tau_2(\tau_2+1)}} + \frac{\sqrt{2\tau_2-2}}{\tau_2} + \sum_{i=1}^{n}2^{i+1}\{\sqrt{\frac{2\tau_2+14i-11}{(\tau_2+7i)(\tau_2+7i+1)}}$$

$$+\sum_{t=1}^{6}\sqrt{\frac{2\tau_2+14i+2t-15}{(\tau_2+7i+t-7)(\tau_2+7i+t-6)}}$$

$$= 2\sqrt{\frac{14n+3}{(7n+2)(7n+3)}} + \frac{\sqrt{14n+2}}{7n+2} + \sum_{i=1}^{n}2^{i+1}\{\sqrt{\frac{14n+14i+3}{(7n+7i+2)(7n+7i+3)}}$$

$$+\sum_{t=1}^{6}\sqrt{\frac{14n+14i+2t-11}{(7n+7i+t-5)(7n+7i+t-4)}}\}$$

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